ISRAEL JOURNAL OF MATHEMATICS 97 (1997), 175–181

HIGHLY NONINTEGRABLE FUNCTIONS IN THE UNIT BALL

BY

PIOTR JAKÓBCZAK*

Uniwersytet Jagielloński, Instytut Matematyki
ul. Reymonta 4, 30 - 059 Kraków
e-mail: jakobcza@im.uj.edu.pl

ABSTRACT

In this note we construct the function f holomorphic in the unit ball B in $\mathbb{C}^N$ such that for every positive-dimensional subspace Π of $\mathbb{C}^N$, $f|_{\Pi\cap B}$ is not L^2-integrable. We present also some possible generalizations of this result.

1. Introduction

Let D be an open set in $\mathbb{C}^N$. Suppose that F is some class of complex-valued functions in D which are holomorphic in D and satisfy some other conditions there. Let M be a linear subvariety of dimension L in $\mathbb{C}^N$. The problem is to determine what further properties (beyond being holomorphic) have the functions from the class F restricted to the slices $L \cap D$. The problem of investigation of holomorphic functions on slices was carried out in many situations by several authors; see e.g. [1], [5], [6], [7].

In [2], [3], and [4] we tried to describe the so-called exceptional subsets for functions from the Bergman spaces. This leads to the general question, how bad the holomorphic function in the domain $D \subset \mathbb{C}^N$ can be, when one considers the L^2-integrability on slices. In this note we prove that for every positive integer N there exists the function f, holomorphic in the unit ball B in $\mathbb{C}^N$, such that for every positive-dimensional subspace Π of $\mathbb{C}^N$, $f|_{\Pi\cap B}$ is not L^2-integrable in $\Pi \cap B$. We present also some possible generalizations of this result.

---

* Partially supported by the KBN Grant 2 PO3A 060 08.
Received March 23, 1995

ACKNOWLEDGEMENT: The author is very indebted to J. Chaumat, M. Jarnicki, and P. Pflug for valuable suggestions and to Aleksandra Palij for helpful discussions.

2. Highly nonintegrable holomorphic functions in the unit ball

In this chapter we prove the following theorem:

THEOREM 1: *Let N be a positive integer. Then there exists the function f holomorphic in the unit ball $B \subset \mathbb{C}^N$, such that for every L-dimensional subspace Π of $\mathbb{C}^N$, $L = 1, \dots, N$,*

$$f|_{\Pi\cap B} \notin L^2(\Pi \cap B)$$

(with respect to the $2L$-dimensional Lebesgue measure in $\Pi \cap B$).

Proof: If D is a domain in $\mathbb{C}^M$, where M is some positive integer, denote by $L^2H(D)$ the space of all functions holomorphic in D and L^2-integrable (with respect to the Lebesgue measure). Let $f \in L^2H(D)$. Applying Cauchy's integral formula for a polydisc, we obtain the well-known estimate: For every $z_0 \in D$,

$$|f(z_0)| \le \left(\frac{M}{\pi}\right)^{M/2} \frac{1}{\operatorname{dist}(z_0, \partial D)^M} \|f\|_{D,2}. \tag{1}$$

(Here and in the sequel, $\|f\|_{D,2}$ denotes the L^2-norm of f in D, and $\operatorname{dist}(z_0, \partial D)$ is the usual euclidean distance of z_0 to the boundary of D.)

If Π is an L-dimensional subspace of $\mathbb{C}^N$, and $f|_{\Pi\cap B} \in L^2(\Pi \cap B)$, then by (1), there exists a constant $c > 0$ such that for every $z \in \Pi \cap B$:

$$|f(z)| \le \frac{cL^{L/2}}{\pi^{L/2} \operatorname{dist}(z, \partial B)^L}.$$

Since

$$\frac{1}{\operatorname{dist}(z, \partial B)^L} \le \frac{1}{\operatorname{dist}(z, \partial B)^N}, \quad L = 1, 2, \dots, N,$$

then it is sufficient to find $f \in \mathcal{O}(B)$ such that for every subspace Π of $\mathbb{C}^N$ of positive dimension, there exists a sequence $\{z_n\}_{n=1}^{\infty} \subset \Pi \cap B$ such that

$$|f(z_n)| \ge \frac{n}{\operatorname{dist}(z_n, \partial B)^N} \quad \text{for infinitely many } n. \tag{2}$$

It follows that it suffices to construct the function $f \in \mathcal{O}(B)$ such that for every $w \in \partial B$, there exists a sequence $\{z_n\}_{n=1}^{\infty}$ of points of the interval $\{tw | 0 \le t \le 1\}$ (the radius of B, joining 0 and w), such that (2) holds.

Every real $(2N-1)$-dimensional hyperplane Θ of $\mathbb{C}^N = \mathbb{R}^{2N}$ with $0 \notin \Theta$ can be represented in the form

$$\Theta = \{z \in \mathbb{C}^N \colon \operatorname{Re} < z - cz_0, z_0 >= 0\}, \tag{3}$$

with some $z_0 \in \mathbb{C}^N, \|z_0\| = 1$, and $c > 0$. ($<\ ,\ >$ denotes here the standard complex scalar product in $\mathbb{C}^N$, and $\|\ \|$ the usual euclidean norm.) Consider the function

$$h(z) = b\exp(a < z - cz_0, z_0 >), \quad z \in \mathbb{C}^N, \quad a, b > 0.$$

Then because of (3), $|\, h|_\Theta \,| \equiv b$, and $|h(z)| < b$ for those $z \in \mathbb{C}^N$ which lie on the same side of the hyperplane Θ as the point 0. (In particular, $|h(0)| = be^{-ac}$.)

Take a sequence $\{r_n\}_{n=1}^\infty$ of positive numbers such that $0 < r_1 < r_2 < \cdots$, and $r_n \nearrow 1$. Then for every $n = 2, 3, \ldots$, there exists a finite number of real hyperplanes in $\mathbb{C}^N$, $\{\Theta_{n,1}, \ldots, \Theta_{n,i_n}\}$, each of them not containing zero, and the family of compact subsets $K_{n,i} \subset \Theta_{n,i}$, $i = 1, \ldots, i_n$, such that:

(4) For every $i = 1, \ldots, i_n$, $K_{n,i} \subset \{z \in \mathbb{C}^N \colon r_{n-1} < \|z\| < r_n\}$.

(5) For every $w \in \partial B$, there exists i with $1 \le i \le i_n$, such that the interval $\{tw \colon 0 \le t \le 1\}$ intersects $K_{n,i}$.

(6) For every $i = 1, \ldots, i_n$, the closed ball $\overline{B(0, r_{n-1})}$ and the sets $K_{n,1}, \ldots, K_{n,i-1}$ lie on the same side of $\Theta_{n,i}$ as the point zero .

(We set here $K_{n,0} = \emptyset$.)

Order the hyperplanes $\Theta_{n,i}$ into the sequence by the following way:

$$\{\Theta_{2,1}, \ldots, \Theta_{2,i_2}, \Theta_{3,1}, \ldots, \Theta_{3,i_3}, \Theta_{4,1}, \ldots\} =: \{\Theta_1, \Theta_2, \ldots\};$$

order similarly the corresponding sets $K_{n,i}$ into the sequence $\{K_1, K_2, \ldots\}$. Every hyperplane $\Theta_{n,i}$ has the form

$$\Theta_{n,i} = \{z \in \mathbb{C}^N \colon \operatorname{Re} < z - c_{n,i}z_{n,i}, z_{n,i} >= 0\}, \quad n = 2, 3, \ldots, \quad i = q, \ldots, i_n,$$

for some $z_{n,i} \in \mathbb{C}^N, \|z_{n,i}\| = 1$, and some $c_{n,i} > 0$. Consider the functions

$$h_{n,i}(z) = b_{n,i}\exp(a_{n,i} < z - c_{n,i}z_{n,i}, z_{n,i} >),$$

where the positive constants $a_{n,i}$ and $b_{n,i}$ are to be determined. In the sequel we assume that the functions $h_{n,i}$ and the constants $a_{n,i}$ and $b_{n,i}$ are ordered into the sequences $\{h_1, h_2, \dots\}$,$\{a_1, a_2, \dots\}$, and $\{b_1, b_2, \dots\}$ in the same way as the hyperplanes $\Theta_{n,i}$ and sets $K_{n,i}$.

Choose $b_1 = b_{2,1}$ in such a way that

$$b_1 = b_{2,1} \geq \frac{1}{\operatorname{dist}(K_{2,1}, \partial B)^N}.$$

Then choose $a_1 = a_{1,2}$ so that

$$\|h_{2,1}\|_{\overline{B(0,r_1)},\infty} \leq 2^{-1}.$$

(This is possible because of (4) and (6); $\|g\|_{L,\infty}$ denotes here the supremum norm of the function g on the set L.) Suppose that the constants $b_1, \dots, b_l$ and $a_1, \dots, a_l$ are already chosen. Then we have $b_{l+1} = b_{n_{l+1},i}$ for uniquely determined n_{l+1} and $1 \leq i \leq i_{n_{l+1}}$. Choose b_{l+1} in such a way that

$$b_{l+1} = b_{n_{l+1},i} \geq \frac{l+1}{\operatorname{dist}(K_{l+1}, \partial B)^N} + \sum_{p=1}^{l} \|h_p\|_{K_{l+1},\infty} + 1. \tag{7}$$

For brevity, denote

$$L_l =: \overline{B(0, r_{n_{l+1}-1})} \cup K_{n_{l+1},1} \cup \dots \cup K_{n_{l+1},i-1}.$$

Then choose $a_{l+1} = a_{n_{l+1},i}$ so that

$$\|h_{l+1}\|_{L_l} (= \|h_{n_{l+1},i}\|_{L_l}) \leq 2^{-(l+1)}. \tag{8}$$

(This is possible because of (6).) Set

$$f(z) = \sum_{l=1}^{\infty} h_l(z), \quad z \in B. \tag{9}$$

It follows from (8), the definition of L_l, and the assumption on the sequence $\{r_n\}$ that the series in the right-hand side of (9) is convergent uniformly on compact subsets of B to some function holomorphic in B. Also, by (7) and (8), for every $z \in K_l$,

$$|f(z)| \geq \frac{l}{\operatorname{dist}(z, \partial B)^N}. \tag{10}$$

By (10) and (5) we conclude that also (2) holds. As explained above, this finishes the proof of the theorem. ∎

3. Possible generalizations

We present here some examples which show that the construction used in the proof of Theorem 1, together with some modifications, allows one to prove the analogues of Theorem 1 in more general situations; the generalizations concern the values of the exponent p in the scale of L^p-spaces, the domains (geometrically convex, or strictly pseudoconvex domains instead of the ball), and the slices (well-behaved analytic subsets of the domain instead of complex subspaces).

Let C be an arbitrary geometrically convex (not necessarily bounded) domain in $\mathbb{C}^N$. Take a sequence of bounded, smoothly bounded, convex domains in $\mathbb{C}^N$ such that $C_1 \subset \bar{C_1} \subset C_2 \subset \bar{C_2} \subset \cdots$, and $\bigcup_{n=1}^{\infty} C_n = C$. We can assume that $0 \in C_1$. Then for every $n = 2, 3, \ldots$, one can find a finite number of real hyperplanes in $\mathbb{C}^N$, $\{\Theta_{n,1}, \ldots, \Theta_{n,i_n}\}$, each of them not containing zero, and the family of compact subsets $K_{n,i} \subset \Theta_{n,i}$, $i = 1, \ldots, i_n$, such that:

(11) $$\text{For every } i = 1, \ldots, i_n, \quad K_{n,i} \subset C_n \setminus \overline{C_{n-1}}.$$

(12) For every $w, u \in \partial C$: if the interval $I_{w,u} =: \{w + t(u-w) \colon 0 \le t \le 1\}$ (joining w and u) intersects $\overline{C_{n-1}}$, then there exists i with $1 \le i \le i_n$ such that $K_{n,i} \cap I_{w,u} \neq \emptyset$.

(13) For every $i = 1, \ldots, i_n$, the sets $\overline{C_{n-1}}$ and $K_{n,1}, \ldots, K_{n,i-1}$ lie on the same side of $\Theta_{n,i}$ as the point zero.

Then, arguing as in the proof of Theorem 1, we can prove that:

For every $w, u \in \partial C$, there exists a sequence $\{z_n\}_{n=1}^{\infty}$ of points of the interval $I_{u,w}$, joining w and u, such that
$$|f(z_n)| \ge \frac{n}{\operatorname{dist}(z_n, \partial C)^N} \quad \text{for infinitely many } n.$$

Therefore, by the explanation in Section 2, we obtain:

THEOREM 2: *If C is a geometrically convex domain in $\mathbb{C}^N$, then there exists a function $f \in \mathcal{O}(D)$ such that for every positive-dimensional affine subspace Π of $\mathbb{C}^N$, if $\Pi \cap C \neq \emptyset$, then*
$$f|_{\Pi \cap C} \notin L^2(\Pi \cap C).$$

Consider now the spaces $L^p, 1 \le p < +\infty$, instead of L^2. It is also well-known that for such p the following estimates, similar to (1), hold: If D is a domain in $\mathbb{C}^M$ for some positive integer M, $f \in L^pH(D)$ (the space of all functions holomorphic and L^p-integrable in D), $1 \le p < +\infty$, and $z_0 \in D$, then

$$|f(z_0)| \le \left(\frac{M}{\pi}\right)^{\frac{M}{p}} \frac{1}{\operatorname{dist}(z_0, \partial D)^{\frac{2M}{p}}} \|f\|_{D,p}. \tag{14}$$

($\|f\|_{D,p}$ denotes the L^p-norm of f in D.) For z_0 sufficiently close to ∂D, we have $\operatorname{dist}(z_0, \partial D) < 1$. Hence for $1 \le p < +\infty$,

$$1 \le \frac{1}{\operatorname{dist}(z_0, \partial D)^{\frac{2M}{p}}} \le \frac{1}{\operatorname{dist}(z_0, \partial D)^{2M}}.$$

Therefore, for all $z_0 \in D$, and for every $1 \le p < +\infty$, we have

$$\frac{1}{\operatorname{dist}(z_0, \partial D)^{\frac{2M}{p}}} \le 1 + \frac{1}{\operatorname{dist}(z_0, \partial D)^{2M}}. \tag{15}$$

Moreover, there exists $c > 0$ such that for all $L = 1, \ldots, N$, and every $1 \le p < +\infty$,

$$\left(\frac{L}{\pi}\right)^{\frac{L}{p}} \le c. \tag{16}$$

It follows from (14), (15), and (16), and the construction from the proof of Theorem 1, that if we require that the function f, constructed similarly as before, satisfies, instead of (10), the inequality

$$|f(z)| \ge \frac{l}{\operatorname{dist}(z_0, \partial D)^{2N}} + 1$$

for all $z \in K_l$ (where the sets K_l are defined as in the proof of Theorem 1), then we obtain the function holomorphic in the unit ball $B \subset \mathbb{C}^N$ such that for every subspace Π of $\mathbb{C}^N$, and for every $1 \le p < +\infty$, $f \notin L^p(\Pi \cap B)$.

The above construction can be carried out also if the ball B is replaced by a convex domain $C \subset \mathbb{C}^N$, and Π is an arbitrary positive-dimensional affine subspace of $\mathbb{C}^N$ such that $\Pi \cap C \neq \emptyset$; the resulting function f is then such that for every $1 \le p < +\infty$,

$$f|_{\Pi \cap C} \notin L^p(\Pi \cap C).$$

Suppose now that C is a strictly convex domain in $\mathbb{C}^N$ with smooth boundary. Then the convenient modification of the above construction allows one to obtain the function holomorphic in C such that $f \notin L^p(\Pi \cap C)$, where Π is now an arbitrary complex submanifold of a neighborhood of $\bar{C}$ such that $\Pi \cap C \neq \emptyset$, and Π intersects ∂C transversally, and p is any number with $1 \leq p < +\infty$.

Finally, let D be a strictly pseudoconvex domain in $\mathbb{C}^N$ with smooth boundary. By Fornaess' embedding theorem, there exists a neighborhood $\tilde{D}$ of $\bar{D}$, a strictly convex, bounded, and smoothly bounded domain C in some $\mathbb{C}^M$, and a holomorphic mapping $\psi\colon \tilde{D} \to \mathbb{C}^M$ such that ψ maps $\tilde{D}$ biholomorphically onto some complex submanifold $\psi(\tilde{D})$ of $\mathbb{C}^M$, such that $\psi(\tilde{D})$ intersects ∂C transversally. If Π is a complex submanifold of some neighborhood of $\tilde{D}$ which intersects ∂D transversally, then $\psi(\Pi)$ is a complex submanifold of some neighborhood of C, also intersecting ∂C transversally. Moreover, $g \in L^p(\psi(\Pi))$ iff $g \circ \psi \in L^P(\Pi)$. By this, and by the above-mentioned result for strictly convex domains, we see that the analogous result holds also for strictly pseudoconvex domains:

THEOREM 3: *Let D be a strictly pseudoconvex domain in $\mathbb{C}^N$ with C^∞ boundary. Then there exists a function f, holomorphic in D, such that for every p with $1 \leq p < +\infty$, and every complex submanifold Π of some neighborhood of $\bar{D}$, such that $\Pi \cap D \neq \emptyset$ and Π intersects ∂D transversally, $f|_{\Pi \cap D} \notin L^p(\Pi \cap D)$.*

References

[1] J. Globevnik and E. L. Stout, *Highly noncontinuable functions on convex domains,* Bulletin des Sciences Mathématiques **104, No 4** (1980), 417–434.

[2] P. Jakóbczak, *The exceptional sets for functions from the Bergman space,* Portugaliae Mathematica **50, No 1** (1993), 115–128.

[3] P. Jakóbczak, *The exceptional sets for functions of the Bergman space in the unit ball,* Rendiconti della Matematica Accademia di Lincei **9, 4** (1993), 79–85.

[4] P. Jakóbczak, *The exceptional sets in the circles for functions from the Bergman space,* Proceedings of the International Symposium Classical Analysis, Kazimierz, 1993, pp. 49–57.

[5] J. Janas, *On a theorem of Lebow and Mlak for several commuting operators,* Studia Mathematica **76** (1983), 249–253.

[6] J. Siciak, *Highly noncontinuable functions on polynomially convex sets,* Zeszyty Naukowe Uniwersytetu Jagiellońskiego **25** (1985), 95–107.

[7] A. Zeriahi, *Ensembles pluripolaires exceptionnels pour la croissance partielle des fonctions holomorphes,* Annales Polonici Mathematici **50** (1989), 81–91.